

# Continuity of Pseudo-Differential Operator $h_{\mu,a}$ Involving Hankel Translation and Hankel Convolution on Some Gevrey Spaces

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## Abstract

The Pseudo-Differential Operator (p.d.o.)  $h_{\mu,a}$  associated with the Bessel Operator involving the symbol  $a(x, y)$  whose derivatives satisfy certain growth conditions depending on some increasing sequences is studied on certain Gevrey spaces. The p.d.o.  $h_{\mu,a}$  on Hankel translation  $\tau$  and Hankel convolution of Gevrey functions is continuous linear map into another Gevrey spaces.

**Key Words:** Hankel transformation, Hankel translation, Hankel convolution, Pseudo-differential operator, Gevrey space.

**MSC:** 46F05, 46F12.

## 1 Introduction

The pseudo-differential operator (p.d.o.)  $h_{\mu,a}$  have applications in the study of boundary value problems on the half line. The p.d.o.  $h_{\mu,a}$  was introduced by [5], and its properties were investigated using Zemanian's theory of the Hankel transformation to certain space of Ultradistributions. Zemanian's theory was further extended by Lee [3], Pathak and Prasad [5], for this purpose, the spaces  $H_{\mu,a_k,A}$ ,  $H_{\mu}^{b_q,B}$  and  $H_{\mu,a_k,A}^{b_q,B}$  of ultradifferentiable function were defined as follows. Similar spaces have been studied in [2] and [7]. Zemanian [8] introduced the function space  $H_{\mu}$  consisting of all complex valued infinitely differentiable functions  $\phi$  defined on  $I = (0, \infty)$  satisfying

$$\gamma_{m,k}^{\mu}(\phi) = \sup_{x \in I} |x^m (x^{-1} d/dx)^k x^{-\mu-1/2} \phi(x)| < \infty, \quad \forall m, k \in \mathbf{N}_0. \quad (1)$$

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The pseudo-differential operator involving the symbol  $a(x, y)$  is defined by

$$(h_{\mu,a}\phi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy) a(x, y) \widehat{\phi}(y) dy, \quad \mu \geq -1/2 \quad (2)$$

where  $\widehat{\phi}$  is the Hankel transformation defined by

$$\widehat{\phi}(y) = (h_\mu \phi)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) \phi(x) dx, \quad (3)$$

and  $J_\mu$  is the Bessel function of the first kind of order  $\mu$ . We shall study the properties of the symbol  $a(x, y)$  in section 2. The following definitions and results will be needed in the sequel.

The space  $L_\mu^p(\mu \geq -1/2)$  is the set of all measurable functions  $\phi$  on  $I = (0, \infty)$  such that

$$\|\phi\|_\mu^p = \int_0^\infty |\phi(x)|^p x^{\mu+1/2} dx < \infty. \quad (4)$$

The Hankel translation of  $\phi \in L_\mu^1(I)$  is defined by

$$(\tau_z \phi)(w) = \int_0^\infty \phi(y) D_\mu(y, w, z) dy, \quad \forall w, z \in I \quad (5)$$

where

$$D_\mu(y, w, z) = \int_0^\infty t^{-\mu-1/2} j_\mu(yt) j_\mu(wt) j_\mu(zt) dt, \quad (6)$$

and

$$j_\mu(wt) = (wt)^{1/2} J_\mu(wt). \quad (7)$$

The Hankel Convolution transform of two functions  $\phi, \psi \in L_\mu^1(I)$  is defined by

$$(\phi \# \psi)(z) = \int_0^\infty \phi(w) (\tau_z \psi)(w) dw, \quad a.e. z \in I \quad (8)$$

we shall also make use of the following results [[1], p. 285]

$$h_\mu(\tau_z \phi)(u) = u^{-\mu-1/2} j_\mu(uz) (h_\mu \phi)(u), \quad \forall u, z \in I \quad (9)$$

and

$$h_\mu(\phi \# \psi)(u) = u^{-\mu-1/2} (h_\mu \phi)(u) (h_\mu \psi)(u). \quad \forall u \in I \quad (10)$$

We shall use the notation and terminology of [4, 5, 8]. The differential operators  $N_\mu, M_\mu$  and  $S_\mu$  are defined by

$$N_\mu = N_{\mu,x} = x^{\mu+1/2}(d/dx)x^{-\mu-1/2}, \quad (11)$$

$$M_\mu = M_{\mu,x} = x^{-\mu-1/2}(d/dx)x^{\mu+1/2}, \quad (12)$$

$$S_\mu = S_{\mu,x} = M_\mu N_\mu = d^2/dx^2 + \frac{(1-4\mu^2)}{4x^2}. \quad (13)$$

We have the following relations for any  $\phi \in H_\mu$  :

$$h_{\mu+1}(-x\phi) = N_\mu h_\mu \phi, \quad (14)$$

$$h_{\mu+1}(N_\mu \phi) = -y h_\mu \phi, \quad (15)$$

$$h_\mu(S_\mu \phi) = -y^2 h_\mu \phi \quad (16)$$

and

$$S_\mu^r \phi(x) = \sum_{j=0}^r b_j x^{2j+\mu+1/2} (d/dx)^{r+j} x^{-\mu-1/2} \phi(x), \quad (17)$$

where the  $b_j$  are constants depending on  $\mu$ .

The following formula are given in [[8],pp.129,134] and [[4],pp.240,242]

$$(x^{-1}d/dx)^k (x^{-\mu-1/2}\psi\phi) = \sum_{\nu=0}^k \binom{k}{\nu} (x^{-1}d/dx)^\nu \psi (x^{-1}d/dx)^{k-\nu} (x^{-\mu-1/2}\phi) \quad (18)$$

$$(x^{-1}d/dx)^k (x^{-\mu} J_\mu(x)) = (-1)^k (x)^{-(\mu+k)} J_{\mu+k}(x) \quad (19)$$

$$(x^{-1}d/dx)^k (x^\mu J_\mu(x)) = (x)^{\mu-k} J_{\mu-k}(x). \quad (20)$$

Let  $\{a_k\}_{k \in \mathbf{N}_0}$  and  $\{b_q\}_{q \in \mathbf{N}_0}$  be arbitrary sequences of positive numbers which satisfy the following conditions

$$a_k^2 \leq a_{k-1} a_{k+1}, \quad \forall k \geq 1 \quad (21)$$

$$b_q^2 \leq b_{q-1} b_{q+1} \quad \forall q \geq 1 \quad (22)$$

Immediate consequences of these inequalities are

$$a_p a_k \leq a_0 a_{p+k}, \quad \forall p, k = 0, 1, 2, \dots \quad (23)$$

$$b_p b_q \leq b_0 b_{p+q}. \quad \forall p, q = 0, 1, 2, \dots \quad (24)$$

from inequality ( 21) it can be proved that

$$(a_k/a_{k+1}) \leq (a_{k-1}/a_k) \leq (a_{k-2}/a_{k-1}) \cdots \leq (a_0/a_1), \quad (25)$$

and

$$\begin{aligned} a_{k-r} &= (a_{k-r}/a_{k-r+1}) \times (a_{k-r+1}/a_{k-r+2}) \times \cdots (a_{k+1}/a_k) \times a_k \\ &\leq (a_0/a_1) \times (a_0/a_1) \times \cdots (a_0/a_1) \times a_k; \end{aligned}$$

so that

$$a_{k-r} \leq (a_0/a_1)^r \times a_k. \quad (26)$$

Furthermore, assume that there are constants  $R_1, R_2 > 0$  and  $H_1, H_2 > 1$  such that

$$a_p \leq R_1 H_1^p \min_{0 \leq q \leq p} a_q a_{p-q}, \quad \forall p, q \in \mathbf{N}_0 \quad (27)$$

$$b_p \leq R_2 H_2^p \min_{0 \leq q \leq p} b_q b_{p-q}, \quad \forall p, q \in \mathbf{N}_0. \quad (28)$$

Let the constants  $c_1, h_1, c_2, h_2, c$  and  $h$  be such that for all  $k, q \in \mathbf{N}_0$ ,

$$a_{k+1} \leq c_1 h_1^k a_k, \quad (29)$$

$$b_{q+1} \leq c_2 h_2^q b_q, \quad (30)$$

$$b_{q+1} \geq c h^q b_q. \quad (31)$$

The conditions ( 29) and ( 30) may be replaced by the following stronger conditions whenever necessary

$$a_{r+k} \leq L_1 R_1^{k+r} a_r a_k, \quad \forall r, k \geq 0 \quad (32)$$

$$b_{r+q} \leq L_2 R_2^{r+q} b_r b_q, \quad \forall r, q \geq 0 \quad (33)$$

where  $L_1, R_1, L_2$  and  $R_2$  are positive constants.

The spaces of type  $H_\mu$ , that is  $H_{\mu, a_k, A}, H_\mu^{b_q, B}$  and  $H_{\mu, a_k, A}^{b_q, B}$  are defined as follows:

**Definition 1.1** Let  $\phi$  be infinitely differentiable function on  $I$ . Then  $\phi \in H_{\mu, a_k, A}$  if and only if

$$\|\phi\|_q^\mu = \sup_{k \in \mathbf{N}_0} \sup_{x \in I} \frac{|x^k (x^{-1} d/dx)^q x^{-\mu-1/2} \phi(x)|}{(A+\sigma)^k a_k} < \infty$$

for every  $q \in \mathbf{N}_0$  where  $A$  is a certain positive constant depending on  $\phi$  and  $\sigma > 0$  is arbitrary.

**Definition 1.2** The space  $H_\mu^{b_q, B}$  is defined as follows:  $\phi \in H_\mu^{b_q, B}$  if and only if

$$\|\phi\|_k^\mu = \sup_{q \in \mathbf{N}_0} \sup_{x \in I} \frac{|x^k (x^{-1} d/dx)^q x^{-\mu-1/2} \phi(x)|}{(B+\rho)^q b_q} < \infty$$

for every  $k \in \mathbf{N}_0$  where  $B$  is a positive constant depending on  $\phi$  and  $\rho > 0$  is arbitrary.

**Definition 1.3** The function  $\phi \in H_{\mu, a_k, A}^{b_q, B}$  if and only if

$$\|\phi\|^\mu = \sup_{k, q \in \mathbf{N}_0} \sup_{x \in I} \frac{|x^k (x^{-1} d/dx)^q x^{-\mu-1/2} \phi(x)|}{(A+\sigma)^k a_k (B+\rho)^q b_q} < \infty$$

where  $\sigma$  and  $\rho$  are as above and  $A$  and  $B$  are certain positive constants depending on  $\phi$ .

The elements of the spaces  $H_{\mu, a_k, A}$ ,  $H_\mu^{b_q, B}$  and  $H_{\mu, a_k, A}^{b_q, B}$  are called Ultradifferentiable functions and those of the corresponding dual spaces  $(H_{\mu, a_k, A})'$ ,  $(H_\mu^{b_q, B})'$  and  $(H_{\mu, a_k, A}^{b_q, B})'$  are called Ultradistributions.

From [6], we have the following results:

**Theorem 1.1** If  $\{a_k\}$  satisfies ( 27) and  $\{b_q\}$  satisfies ( 28)  $\forall k, q \in \mathbf{N}_0$ , then for each fixed  $z$ ,  $0 < z < z_0$ ,  $\mu \geq -1/2$ , the mapping  $\phi \mapsto \tau_z \phi$  is continuous from the spaces

(i)  $H_{\mu, a_k, A}^{b_q, B}$  into  $H_{\mu, a_k^3 b_k, A_3}^{a_q^2 b_q^2, B_4}$ , where  $A_3 = A_1 B_3 (R^\otimes)^2$ ,  $B_3 = R_1^2 [B_1 + (z_0 a_0 / a_1)^2]$ ,  $A_1 = AB (R^*)^2$ ,  $B_4 = A_1^2 (R^\otimes)^6$ ,  $R^\otimes = \max(1, R_1 R_2)$  and  $R_1, R_2$  are defined by ( 32) and ( 33),

and

(ii)  $H_{\mu, a_k, A}$  into  $H_{\mu, a_k^2, A_2}$ , where  $A_2 = R_1^2 [B_1 + (z_0 a_0 / a_1)^2]$ ,  $B_1 = A^2 (R^*)^6$  and  $R^* = \max(1, R_1)$ .

We note that the Hankel translation cannot be defined on the whole of the space  $H_\mu^{b_q, B}$ ; but it could be defined on a certain subspace  $\tilde{H}_\mu^{b_q, B}$  of  $H_\mu^{b_q, B}$  in which the following condition is satisfied

$$\sup_k Q_{k+2q}^\mu = Q_q^{*\mu}, \quad (34)$$

where  $Q_q^{*\mu}$  are constants restraining the  $\phi$ 's in  $H_\mu^{b_q, B}$ . Then

$$(iii) \tilde{H}_\mu^{b_q, B} \text{ into } H_\mu^{b_q^2, B_2}, \text{ where } B_2 = B^2(R^*)^6, R^* = \max(1, R_1).$$

## 2 Pseudo-Differential Operator Involving Hankel Translation on the Spaces of Type $H_\mu$

This section investigates the p.d.o. involving Hankel translation  $\tau$  on the spaces  $H_{\mu, a_k, A}$ ,  $H_\mu^{b_q, B}$  and  $H_{\mu, a_k, A}$ .

**Definition 2.1** The symbol  $a(x, y)$  is defined to be a complex valued function belonging to the space  $\mathbf{C}^\infty(I \times I)$ , such that its derivatives satisfy the growth condition

$$|(x^{-1}d/dx)^\alpha (y^{-1}d/dy)^\nu a(x, y)| \leq L_m (C + \delta)^\alpha c_\alpha (D + \eta)^\nu d_\nu (1 + y)^{m-\nu} \quad (35)$$

for all  $\alpha, \nu \in \mathbf{N}_0$ ,  $\delta > 0$ ,  $\eta > 0$  and  $L_m > 0$ , where  $m$  is a fixed real number, and  $\{c_\alpha\}$  and  $\{d_\nu\}$  are certain sequences of positive real numbers satisfying some of the conditions of type ( 21)-( 31). The set of all such symbols will be denoted  $S_{c_\alpha, d_\nu}^m$ .

We have the following interesting results [6].

**Theorem 2.1** If  $\{a_k\}$  and  $\{b_q\} \forall k, q \in \mathbf{N}_0$ , satisfies ( 21) and ( 22) respectively then for each fixed  $z, 0 < z < z_0$  and  $\mu \geq -1/2$ , the mapping  $\phi \mapsto h_\mu \tau_z \phi$  is linear and continuous from

$$(i) H_{\mu, a_k, A} \text{ into } H_\mu^{a_q^2, B_3}, \text{ where } B_3 = R_1^2 [B_1 + (z_0 a_0 / a_1)^2]$$

$$(ii) \tilde{H}_\mu^{b_q, B} \text{ into } H_{\mu, b_k, B},$$

$$(iii) H_{\mu, a_k, A}^{b_q, B} \text{ into } H_{\mu, a_k b_k, A_1}^{a_q^2, B_3}, \text{ where } A_1 = ABH_1^2, B_3 = R_1^2 [B_1 + (z_0 a_0 / a_1)^2] \text{ and } B_1 \text{ as above.}$$

**Theorem 2.2** Let  $\{a_k\}, \{b_k\}, \{c_k\}$  and  $\{d_k\}, k \in \mathbf{N}_0$ , satisfy condition ( 26),  $\mu \geq -1/2$ , and let the symbol  $a(x, y)$  satisfy ( 35) then the p.d.o.  $\phi \mapsto h_{\mu, a} \tau_z \phi$  is continuous linear mapping from  $H_{\mu, a_k, A}^{b_q, B}$  into  $H_{\mu, a_k^* b_k^*, A_6}^{a_q^{*2} b_q^{*2}, B_6}$ , where  $a_k^* = \max_{k \in \mathbf{N}_0} (a_k, d_k), b_q^* = \max_{q \in \mathbf{N}_0} (b_q, c_q), A_6 = ((a_0^*/a_1^*)D + B_3)A_1, B_6 = (b_0^* a_0^{*2} / b_1^* a_1^{*2})C + H^6 A_1^2$  and  $B_3, A_1$  as above.

**Proof:** Suppose that  $\phi \in H_{\mu, a_k, A}^{b_q, B}$ , then by Theorem (2.1)(iii)  $(h_{\mu} \tau_z \phi) \in H_{\mu, a_k b_k, A_1}^{a_q^2, B_3}$ , where  $A_1 = AB(R^*)^2$  and  $B_3 = R_1^2 [B_1 + (z_0 a_0 / a_1)^2], R^* = \max(1, R_1)$ .

Now assume that

$$\begin{aligned} \Phi(x) &= (h_{\mu, a} \tau_z \phi)(x) \\ &= \int_0^\infty (xy)^{1/2} J_\mu(xy) a(x, y) (h_{\mu} \tau_z \phi)(y) dy. \end{aligned}$$

Using Zemanian's technique [[8], p. 144] we have

$$\begin{aligned} N_\mu \Phi(x) &= x^{\mu+1/2} (d/dx) x^{-\mu-1/2} \Phi(x) \\ &= x^{\mu+1+1/2} (x^{-1} d/dx) x^{-\mu-1/2} \Phi(x). \end{aligned} \quad (36)$$

$$\begin{aligned} N_{\mu+1} N_\mu \Phi(x) &= x^{\mu+1+1/2} (d/dx) x^{-(\mu+1)-1/2} N_\mu \Phi(x) \\ &= x^{\mu+2+1/2} (x^{-1} d/dx) x^{-\mu-3/2} [x^{\mu+3/2} (x^{-1} d/dx) x^{-\mu-1/2} \Phi(x)] \\ &= x^{\mu+2+1/2} (x^{-1} d/dx)^2 x^{-\mu-1/2} \Phi(x). \end{aligned}$$

Similary, using ( 18), we have

$$\begin{aligned} N_{\mu+q-1} \dots N_\mu \Phi(x) &= x^{\mu+q+1/2} (x^{-1} d/dx)^q x^{-\mu-1/2} \Phi(x) \\ &= x^{\mu+q+1/2} (x^{-1} d/dx)^q x^{-\mu-1/2} \\ &\quad \times \int_0^\infty (xy)^{1/2} J_\mu(xy) a(x, y) (h_{\mu} \tau_z \phi)(y) dy \\ &= x^{\mu+q+1/2} (x^{-1} d/dx)^q \int_0^\infty y^{1/2} x^{-\mu} J_\mu(xy) a(x, y) (h_{\mu} \tau_z \phi)(y) dy \\ &= x^{\mu+q+1/2} \int_0^\infty y^{1/2} \sum_{r=0}^q \binom{q}{r} (x^{-1} d/dx)^{q-r} x^{-\mu} J_\mu(xy) \\ &\quad \times (x^{-1} d/dx)^r a(x, y) (h_{\mu} \tau_z \phi)(y) dy \\ &= x^{\mu+q+1/2} \int_0^\infty y^{1/2} \sum_{r=0}^q \binom{q}{r} (-y)^{q-r} x^{-\mu-q+r} J_{\mu+q-r}(xy) \\ &\quad \times (x^{-1} d/dx)^r a(x, y) (h_{\mu} \tau_z \phi)(y) dy. \end{aligned} \quad (37)$$

Therefore,

$$\begin{aligned}
N_{\mu+q-1} \dots N_{\mu} \Phi(x) &= \sum_{r=0}^q \binom{q}{r} \int_0^{\infty} x^{r+1/2} y^{1/2} (x^{-1} d/dx)^r a(x, y) \\
&\times (h_{\mu} \tau_z \phi)(y) (-y)^{q-r} J_{\mu+q-r}(xy) dy \quad (38) \\
&= \sum_{r=0}^q \binom{q}{r} x^r \int_0^{\infty} (xy)^{1/2} J_{\mu+q-r}(xy) \\
&\times [(x^{-1} d/dx)^r a(x, y) (-y)^{q-r} (h_{\mu} \tau_z \phi)(y)] dy. \\
&= \sum_{r=0}^q \binom{q}{r} x^r h_{\mu+q-r} \\
&\times [(x^{-1} d/dx)^r a(x, y) (-y)^{q-r} (h_{\mu} \tau_z \phi)(y)](x). \quad (39)
\end{aligned}$$

Using formula  $-x h_{\mu} \phi = h_{\mu+1}(N_{\mu} \phi)$  in (39), we get

$$\begin{aligned}
(-x) N_{\mu+q-1} \dots N_{\mu} \Phi(x) &= \sum_{r=0}^q \binom{q}{r} x^r (-x) h_{\mu+q-r} [(x^{-1} d/dx)^r a(x, y) (-y)^{q-r} (h_{\mu} \tau_z \phi)(y)](x) \\
&= \sum_{r=0}^q \binom{q}{r} x^r h_{\mu+q-r+1} N_{\mu+q-r} \\
&\times [(x^{-1} d/dx)^r a(x, y) (-y)^{q-r} (h_{\mu} \tau_z \phi)(y)](x) \\
&= \sum_{r=0}^q \binom{q}{r} x^r (-x) \int_0^{\infty} (xy)^{1/2} J_{\mu+q-r+1}(xy) N_{\mu+q-r} \\
&\times [(x^{-1} d/dx)^r a(x, y) (-y)^{q-r} (h_{\mu} \tau_z \phi)(y)] dy \\
&= \sum_{r=0}^q \binom{q}{r} \int_0^{\infty} x^r (xy)^{1/2} J_{\mu+q-r+1}(xy) y^{\mu+q-r+1/2} (d/dy) \\
&\times y^{-\mu-q+r-1/2} [(x^{-1} d/dx)^r a(x, y) (-y)^{q-r} (h_{\mu} \tau_z \phi)(y)] dy \\
&= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} \int_0^{\infty} x^{r+1/2} y^{\mu+q-r+2} (y^{-1} d/dy) \\
&\times [y^{-\mu-1/2} (h_{\mu} \tau_z \phi)(y) (x^{-1} d/dx)^r a(x, y) J_{\mu+q-r+1}(xy)] dy \quad (40)
\end{aligned}$$



$$\begin{aligned}
&= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} x^r \int_0^\infty (xy)^{1/2} J_{\mu+q-r+1}(xy) \\
&\times [y^{\mu+q-r+1+1/2} (y^{-1} d/dy) \{y^{-\mu-1/2} (h_\mu \tau_z \phi)(y) (x^{-1} d/dx)^r a(x, y)\}] dy \\
&= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} x^r h_{\mu+q-r+1} [y^{\mu+q-r+1+1/2} \\
&\times (y^{-1} d/dy) \{y^{-\mu-1/2} (h_\mu \tau_z \phi)(y) (x^{-1} d/dx)^r a(x, y)\}]. \tag{41}
\end{aligned}$$

Now, from (41) again using result  $-x h_\mu \phi = h_{\mu+1}(N_\mu \phi)$ , we get

$$\begin{aligned}
(-x)^2 (N_{\mu+q-1} \dots N_\mu \Phi(x)) &= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} x^r (-x) h_{\mu+q-r+1} \\
&\times [y^{\mu+q-r+1+1/2} (y^{-1} d/dy) \{y^{-\mu-1/2} (h_\mu \tau_z \phi)(y) (x^{-1} d/dx)^r a(x, y)\}] \\
&= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} x^r h_{\mu+q-r+2} N_{\mu+q-r+1} \\
&\times [y^{\mu+q-r+1+1/2} (y^{-1} d/dy) \{y^{-\mu-1/2} (h_\mu \tau_z \phi)(y) (x^{-1} d/dx)^r a(x, y)\}] \\
&= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} x^r \int_0^\infty (xy)^{1/2} J_{\mu+q-r+2}(xy) N_{\mu+q-r+1} \\
&\times [y^{\mu+q-r+1+1/2} (y^{-1} d/dy) \{y^{-\mu-1/2} (h_\mu \tau_z \phi)(y) (x^{-1} d/dx)^r a(x, y)\}] \\
&= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} \int_0^\infty x^{r+1/2} y^{\mu+q-r+2+1} \\
&\times [(y^{-1} d/dy)^2 \{y^{-\mu-1/2} (h_\mu \tau_z \phi)(y) (x^{-1} d/dx)^r a(x, y)\}] \\
&\times J_{\mu+q-r+2}(xy) dy.
\end{aligned}$$

In general, we have

$$\begin{aligned}
(-x)^k (N_{\mu+q-1} \dots N_\mu \Phi(x)) &= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} \int_0^\infty x^{r+1/2} y^{\mu+q-r+k+1} \\
&\times [(y^{-1} d/dy)^k \{y^{-\mu-1/2} (h_\mu \tau_z \phi)(y) (x^{-1} d/dx)^r a(x, y)\}] \\
&\times J_{\mu+q-r+k}(xy) dy. \\
&= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} \int_0^\infty x^{r+1/2} y^{\mu+q-r+k+1} \\
&\times \sum_{\nu=0}^k \binom{k}{\nu} (y^{-1} d/dy)^\nu (x^{-1} d/dx)^r \\
&\times a(x, y) (y^{-1} d/dy)^{k-\nu} y^{-\mu-1/2} (h_\mu \tau_z \phi)(y) J_{\mu+q-r+k}(xy) dy. \tag{42}
\end{aligned}$$

Now, from ( 37), we know that

$$N_{\mu+q-1} \dots N_{\mu} \Phi(x) = x^{\mu+q+1/2} (x^{-1} d/dx)^q x^{-\mu-1/2} \Phi(x). \quad (43)$$

Multiplying both sides in ( 43) by  $(-x)^k$ , we get

$$(-x)^k (N_{\mu+q-1} \dots N_{\mu} \Phi(x)) = (-1)^k x^{\mu+k+q+1/2} (x^{-1} d/dx)^q x^{-\mu-1/2} \Phi(x). \quad (44)$$

Comparing equations ( 42) and ( 44), we have

$$\begin{aligned} (-1)^k x^{\mu+k+q+1/2} (x^{-1} d/dx)^q x^{-\mu-1/2} \Phi(x) &= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} \int_0^{\infty} x^{r+1/2} y^{\mu+q-r+k+1} \\ &\times \sum_{\nu=0}^k \binom{k}{\nu} (y^{-1} d/dy)^{\nu} (x^{-1} d/dx)^r a(x, y) \\ &\times (y^{-1} d/dy)^{k-\nu} y^{-\mu-1/2} (h_{\mu} \tau_z \phi)(y) J_{\mu+q-r+k}(xy) dy. \end{aligned}$$

Therefore,

$$\begin{aligned} (-1)^k x^k (x^{-1} d/dx)^q x^{-\mu-1/2} \Phi(x) &= \sum_{r=0}^q \binom{q}{r} (-1)^{q-r} \int_0^{\infty} x^{-(\mu+q-r)} y^{\mu+q-r+k+1} \\ &\times \sum_{\nu=0}^k \binom{k}{\nu} (y^{-1} d/dy)^{\nu} (x^{-1} d/dx)^r a(x, y) \\ &\times (y^{-1} d/dy)^{k-\nu} y^{-\mu-1/2} (h_{\mu} \tau_z \phi)(y) J_{\mu+q-r+k}(xy) dy. \end{aligned}$$

Thus

$$\begin{aligned} |x^k (x^{-1} d/dx)^q x^{-\mu-1/2} \Phi(x)| &\leq \sum_{r=0}^q \binom{q}{r} \int_0^{\infty} y^{2(\mu+q-r)+k+1} \sum_{\nu=0}^k \binom{k}{\nu} \\ &\times |(x^{-1} d/dx)^r (y^{-1} d/dy)^{\nu} a(x, y)| \\ &\times |(y^{-1} d/dy)^{k-\nu} y^{-\mu-1/2} (h_{\mu} \tau_z \phi)(y)| \\ &\times |(xy)^{-(\mu+q-r)} J_{\mu+q-r+k}(xy)| dy. \end{aligned}$$

Using inequality ( 35), the right-hand side assumes the form

$$\begin{aligned} \sum_{r=0}^q \binom{q}{r} \int_0^{\infty} y^{2(\mu+q-r)+k+1} \sum_{\nu=0}^k \binom{k}{\nu} L_m(C + \delta)^r c_r(D + \eta)^{\nu} d_{\nu} (1 + y)^{m-\nu} \\ \times |(y^{-1} d/dy)^{k-\nu} y^{-\mu-1/2} (h_{\mu} \tau_z \phi)(y)| \frac{2^{-(\mu+q-r)} E}{\Gamma(\mu + q - r + 1)} dy. \end{aligned}$$

If we assume that  $p$  is a positive integer such that  $p \geq m$  and  $s > 2\mu + 1$ , then the last term can be estimated by

$$\begin{aligned}
|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)| &\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} \sum_{r=0}^q \sum_{\nu=0}^k \binom{q}{r} \binom{k}{\nu} (C+\delta)^r c_r (D+\eta)^\nu d_\nu \\
&\times \int_0^\infty y^{2(\mu+q-r)+k+1} (1+y)^{m-\nu+s} \\
&\times |(y^{-1}d/dy)^{k-\nu} y^{-\mu-1/2} (h_\mu \tau_z \phi)(y)| \frac{dy}{(1+y)^s} \\
&\leq \frac{2^{-\mu}EL_m}{\Gamma\mu+1} \sum_{r=0}^q \sum_{\nu=0}^k \binom{q}{r} \binom{k}{\nu} (C+\delta)^r c_r (D+\eta)^\nu d_\nu \\
&\times \sup_{y \in I} [(1+y)^{p+s} y^{2(q-r)+k} |(y^{-1}d/dy)^{k-\nu} y^{-\mu-1/2} (h_\mu \tau_z \phi)(y)|] \\
&\times \int_0^\infty \frac{y^{2(\mu+1/2)}}{(1+y)^s} dy.
\end{aligned}$$

$$\begin{aligned}
|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)| &\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} \sum_{r=0}^q \sum_{\nu=0}^k \binom{q}{r} \binom{k}{\nu} (C+\delta)^r c_r (D+\eta)^\nu d_\nu \\
&\times \sum_{n=0}^{p+s} \binom{p+s}{n} \sup_{y \in I} [y^n y^{2(q-r)+k} \\
&\times |(y^{-1}d/dy)^{k-\nu} y^{-\mu-1/2} (h_\mu \tau_z \phi)(y)|]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} \sum_{r=0}^q \sum_{\nu=0}^k \sum_{n=0}^{p+s} \binom{q}{r} \binom{k}{\nu} \binom{p+s}{n} \\
&\times (C+\delta)^r c_r (D+\eta)^\nu d_\nu \\
&\times \sup_{y \in I} |y^{n+2(q-r)+k} (y^{-1}d/dy)^{k-\nu} y^{-\mu-1/2} (h_\mu \tau_z \phi)(y)|. \quad (45)
\end{aligned}$$

Using Theorem 2.1 (iii) in (45), we have

$$\begin{aligned}
|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)| &\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} \sum_{r=0}^q \sum_{\nu=0}^k \sum_{n=0}^{p+s} \binom{q}{r} \binom{k}{\nu} \binom{p+s}{n} \\
&\times (C+\delta)^r c_r (D+\eta)^\nu d_\nu \|(h_\mu \tau_z \phi)(y)\|^\mu (A_1 + \sigma)^{n+2(q-r)+k} \\
&\times a_{n+2(q-r)+k} b_{n+2(q-r)+k} (B_3 + \rho)^{k-\nu} a_{k-\nu}^2.
\end{aligned}$$

Using inequality ( 27) and ( 28), we have

$$\begin{aligned}
|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)| &\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} \sum_{r=0}^q \sum_{\nu=0}^k \sum_{n=0}^{p+s} \binom{q}{r} \binom{k}{\nu} \binom{p+s}{n} (C+\delta)^r c_r (D+\eta)^\nu d_\nu \\
&\times \|(h_\mu \tau_z \phi)\|^\mu (A_1 + \sigma)^{n+2(q-r)+k} (B_3 + \rho)^{k-\nu} R_1^3 H_1^{n+6(q-r)+2k} \\
&\times a_n a_{q-r}^2 a_k R_2^3 H_2^{n+6(q-r)+2k} b_n b_{q-r}^2 b_k a_{k-\nu}^2.
\end{aligned}$$

Applying inequality ( 23) and ( 24), we have

$$\begin{aligned}
|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)| &\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} (R_1 R_2)^3 \sum_{r=0}^q \sum_{\nu=0}^k \sum_{n=0}^{p+s} \binom{q}{r} \binom{k}{\nu} \binom{p+s}{n} (C+\delta)^r c_r \\
&\times (D+\eta)^\nu d_\nu H^{n+6(q-r)+2k} (A_1 + \sigma)^{n+2(q-r)+k} a_n^* a_k^* a_{q-r}^{*2} \\
&\times (a_{k-\nu}^* a_\nu^*) a_{k-\nu}^* b_n^* b_{q-r}^* (b_{q-r}^* b_r^*) (B_3 + \rho)^{k-\nu} b_k^* \|(h_\mu \tau_z \phi)\|^\mu
\end{aligned}$$

where  $b_q^* = \max_{q \in \mathbf{N}_0} (b_q, c_q)$ ,  $a_k^* = \max_{k \in \mathbf{N}_0} (a_k, d_k)$ ,  $H = H_1 H_2$  and  $R_3 = R_1 R_2$ .

Finally

$$\begin{aligned}
|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)| &\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} (R_3)^3 \sum_{r=0}^q \sum_{\nu=0}^k \sum_{n=0}^{p+s} \binom{q}{r} \binom{k}{\nu} \binom{p+s}{n} (C+\delta)^r \\
&\times H^{n+6(q-r)+2k} (D+\eta)^\nu (A_1 + \sigma)^{n+2(q-r)+k} (B_3 + \rho)^{k-\nu} a_n^* b_n^* b_k^* \\
&\times (a_0^*/a_1^*)^{2r} a_q^{*2} (a_0^*/a_1^*)^\nu a_0^* a_k^* (b_0^*/b_1^*)^r b_q^* b_0^* b_q^* \|(h_\mu \tau_z \phi)\|^\mu \\
&\leq (R_3)^3 \sum_{r=0}^q \binom{q}{r} [(b_0^* a_0^{*2}/b_1^* a_1^{*2})(C+\delta)]^r [H^6 (A_1 + \sigma)^2]^{q-r} \\
&\times a_q^{*2} b_q^{*2} \sum_{\nu=0}^k \binom{k}{\nu} (B_3 + \rho)^{k-\nu} [(a_0^*/a_1^*)(D+\eta)]^\nu (A_1 + \sigma)^k a_k^{*3} b_k^* \|(h_\mu \tau_z \phi)\|^\mu \\
&\leq R (A_6 + \sigma_1)^k a_k^{*3} b_k^* (B_6 + \rho_1)^q a_q^{*2} b_q^{*2} \|(h_\mu \tau_z \phi)\|^\mu
\end{aligned}$$

where  $A_6 = ((a_0^*/a_1^*)D + B_3)A_1$ ,  $B_6 = (b_0^* a_0^{*2}/b_1^* a_1^{*2})C + H^6 A_1^2$  and  $R$  is a constant. Therefore

$$\|\Phi\|^\mu = \sup_{k,q \in \mathbf{N}_0} \sup_{x \in I} \frac{|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)|}{(A_6 + \sigma_1)^k a_k^{*3} b_k^* (B_6 + \rho_1)^q a_q^{*2} b_q^{*2}} \leq R \|(h_\mu \tau_z \phi)\|^\mu.$$

Hence  $\Phi(x) \in H_{\mu, a_k^{*3} b_k^*, A_6}^{a_q^{*2} b_q^{*2}, B_6}$ .

**REMARK:** In Theorem 2.2 we may choose  $\phi \in H_{\mu, a_k, A}(\text{or } \widetilde{H}_\mu^{b_q, B})$  then the p.d.o.  $h_{\mu, a}$  on Hankel translation  $(h_{\mu, a}(\tau_z \phi)) \in \widetilde{H}_{\mu, a_k^*, A'_6}(\text{or } \widetilde{H}_\mu^{b_q^{*2}, B'_6})$  where  $A'_6 = (a_0^*/a_1^*)D + B_3$  and  $B'_6 = (b_0^*/b_1^*) + H_2^6 B^2$ .

### 3 Pseudo-Differential Operator Involving Hankel Convolution on the Spaces of Type $H_\mu$

In this section we investigate the p.d.o. involving Hankel convolution transform of  $\phi\#\psi$  on the spaces  $H_{\mu,a_k,A}$ ,  $H_\mu^{b_q,B}$  and  $H_{\mu,a_k,A}^{b_q,B}$ . We have the following interesting results [6].

**Theorem 3.1** *If  $\{a_k\}$  and  $\{b_q\} \forall k, q \in \mathbf{N}_0$  satisfies ( 21) and ( 22) respectively then for  $\mu \geq -1/2$ , the mapping  $(\phi, \psi) \mapsto (\phi\#\psi)$  is linear and continuous from the spaces*

- (i)  $H_{\mu,a_k,A} \times H_{\mu,a_k,A}$  into  $\tilde{H}_{\mu,a_k^2,B_1}$ , where  $B_1 = A^2(R^*)^6$
- (ii)  $\tilde{H}_\mu^{b_q,B} \times \tilde{H}_\mu^{b_q,B}$  into  $H_\mu^{b_q^2,B_2}$ , where  $B_2 = B^2(R^*)^6$
- (iii)  $H_{\mu,a_k,A}^{b_q,B} \times H_{\mu,a_k,A}^{b_q,B}$  into  $H_{\mu,a_k^3b_k,A_4}^{a_q^2b_q^2,B_5}$ , where  $A_4 = (R^\otimes)^2A_1B_1$ ,  $B_5 = (R^\otimes)^6A_1^2$ , and  $R^\otimes = \max(1, R_1R_2)$ .

**Theorem 3.2** *If  $\{a_k\}$  and  $\{b_q\} \forall k, q \in \mathbf{N}_0$  satisfies ( 21) and ( 22) respectively then for  $\mu \geq -1/2$ , the mapping  $(\phi, \psi) \mapsto h_\mu(\phi\#\psi)$  is continuous linear mapping from*

- (i)  $H_{\mu,a_k,A} \times H_{\mu,a_k,A}$  into  $H_\mu^{a_q^2,B_1}$ , where  $B_1 = A^2(R^*)^6$  and  $R^* = \max(1, R_1)$
- (ii)  $\tilde{H}_\mu^{b_q,B} \times \tilde{H}_\mu^{b_q,B}$  into  $H_{\mu,b_k,B}$ , and
- (iii)  $H_{\mu,a_k,A}^{b_q,B} \times H_{\mu,a_k,A}^{b_q,B}$  into  $H_{\mu,a_kb_k,A_1}^{a_q^2,B_1}$ , where  $A_1 = AB(R^*)^2$ ,  $B_1 = A^2(R^*)^6$  and  $R^* = \max(1, R_1)$ .

**Theorem 3.3** *Let  $\{a_k\}, \{b_k\}, \{c_k\}$  and  $\{d_k\}, k \in \mathbf{N}_0$ , satisfy condition ( 26),  $\mu \geq -1/2$  and the symbol  $a(x, y)$  satisfy ( 35) then the p.d.o  $(\phi, \psi) \mapsto h_{\mu,a}(\phi\#\psi)$  is continuous linear mapping from  $H_{\mu,a_k,A}^{b_q,B} \times H_{\mu,a_k,A}^{b_q,B}$  into  $H_{\mu,a_k^3b_k^*,A_7}^{a_q^{*2}b_q^{*2},B_6}$ , where  $A_7 = ((a_0^*/a_1^*)D + B_1)A_1$  and  $B_6 = (b_0^*a_0^{*2}/b_1^*a_1^{*2})C + H^6A_1^2$ .*

**Proof:** Let  $(\phi, \psi) \in H_{\mu,a_k,A}^{b_q,B} \times H_{\mu,a_k,A}^{b_q,B}$  then by, Theorem 3.2 (iii),  $h_\mu(\phi\#\psi) \in H_{\mu,a_kb_k,A_1}^{a_q^2,B_1}$ .

Now assume that  $\Psi(x) = h_{\mu,a}(\phi\#\psi)(x)$ , from inequality ( 45), we have

$$\begin{aligned} |x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Psi(x)| &\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} \sum_{r=0}^q \sum_{\nu=0}^k \sum_{n=0}^{p+s} \binom{q}{r} \binom{k}{\nu} \binom{p+s}{n} \\ &\times (C+\delta)^r c_r (D+\eta)^\nu d_\nu \\ &\times \sup_{y \in I} |y^{n+2(q-r)+k} (y^{-1}d/dy)^{k-\nu} y^{-\mu-1/2} h_\mu(\phi\#\psi)(y)|. \end{aligned}$$

Using Theorem 3.2 (iii), the right-hand side assumes the form

$$\begin{aligned} |x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Psi(x)| &\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} \sum_{r=0}^q \sum_{\nu=0}^k \sum_{n=0}^{p+s} \binom{q}{r} \binom{k}{\nu} \binom{p+s}{n} \\ &\times (C+\delta)^r c_r (D+\eta)^\nu d_\nu \|h_\mu(\phi\#\psi)\|^\mu (A_1+\sigma)^{n+2(q-r)+k} \\ &\times a_{n+2(q-r)+k} b_{n+2(q-r)+k} (B_1+\rho)^{k-\nu} a_{k-\nu}^2. \end{aligned}$$

Using the inequalities ( 27) and ( 28) we can bound this expression by

$$\begin{aligned} \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} \sum_{r=0}^q \sum_{\nu=0}^k \sum_{n=0}^{p+s} \binom{q}{r} \binom{k}{\nu} \binom{p+s}{n} (C+\delta)^r c_r (D+\eta)^\nu d_\nu \|h_\mu(\phi\#\psi)\|^\mu (A_1+\sigma)^{n+2(q-r)+k} \\ \times (B_1+\rho)^{k-\nu} R_1^3 H_1^{n+6(q-r)+2k} a_n a_{q-r}^2 a_k R_2^3 H_2^{n+6(q-r)+2k} b_n b_{q-r}^2 b_k a_{k-\nu}^2. \end{aligned}$$

Applying inequality ( 23) and ( 24), we have

$$\begin{aligned} |x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Psi(x)| &\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} (R_1 R_2)^3 \sum_{r=0}^q \sum_{\nu=0}^k \sum_{n=0}^{p+s} \binom{q}{r} \binom{k}{\nu} \binom{p+s}{n} (C+\delta)^r c_r \\ &\times (D+\eta)^\nu d_\nu H^{n+6(q-r)+2k} (A_1+\sigma)^{n+2(q-r)+k} a_n^* a_k^* a_{q-r}^{*2} \\ &\times (a_{k-\nu}^* a_\nu^*) a_{k-\nu}^* b_n^* b_{q-r}^* (b_{q-r}^* b_r^*) (B_1+\rho)^{k-\nu} b_k^* \|h_\mu(\phi\#\psi)\|^\mu, \end{aligned}$$

where  $b_q^* = \max_{q \in \mathbf{N}_0} (b_q, c_q)$ ,  $a_k^* = \max_{k \in \mathbf{N}_0} (a_k, d_k)$ ,  $H = H_1 H_2$  and  $R_3 = R_1 R_2$ .

Finally

$$\begin{aligned} |x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Psi(x)| &\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)} (R_3)^3 \sum_{r=0}^q \sum_{\nu=0}^k \sum_{n=0}^{p+s} \binom{q}{r} \binom{k}{\nu} \binom{p+s}{n} (C+\delta)^r \\ &\times H^{n+6(q-r)+2k} (D+\eta)^\nu (A_1+\sigma)^{n+2(q-r)+k} (B_1+\rho)^{k-\nu} a_n^* b_n^* b_k^* \\ &\times (a_0^*/a_1^*)^{2r} a_q^{*2} (a_0^*/a_1^*)^\nu a_0^* a_k^* (b_0^*/b_1^*)^r b_q^* b_q^* \|h_\mu(\phi\#\psi)\|^\mu \end{aligned}$$

$$\begin{aligned}
&\leq (R_3)^3 \sum_{r=0}^q \binom{q}{r} [(b_0^* a_0^{*2} / b_1^* a_1^{*2})(C + \delta)]^r [H^6(A_1 + \sigma)^2]^{q-r} \\
&\times a_q^{*2} b_q^{*2} \sum_{\nu=0}^k \binom{k}{\nu} (B_1 + \rho)^{k-\nu} [(a_0^* / a_1^*)(D + \eta)]^\nu (A_1 + \sigma)^k a_k^{*3} b_k^* \|h_\mu(\phi \# \psi)\|^\mu \\
&\leq R(A_7 + \sigma_1)^k a_k^{*3} b_k^* (B_6 + \rho_1)^q a_q^{*2} b_q^{*2} \|h_\mu(\phi \# \psi)\|^\mu,
\end{aligned}$$

where  $A_7 = ((a_0^* / a_1^*)D + B_1)A_1$ ,  $B_6 = (b_0^* a_0^{*2} / b_1^* a_1^{*2})C + H^6 A_1^2$  and  $R$  is a constant. Therefore

$$\|\Psi\|^\mu = \sup_{k,q \in \mathbf{N}_0} \sup_{x \in I} \frac{|x^k (x^{-1} d/dx)^q x^{-\mu-1/2} \Psi(x)|}{(A_7 + \sigma_1)^k a_k^{*3} b_k^* (B_6 + \rho_1)^q a_q^{*2} b_q^{*2}} \leq R \|h_\mu(\phi \# \psi)\|^\mu.$$

Hence  $\Psi(x) \in H_{\mu, a_k^{*3} b_k^*, A_7}^{a_q^{*2} b_q^{*2}, B_6}$ .

**REMARK:** In Theorem 3.3 we may also choose  $\phi, \psi \in H_{\mu, a_k, A}(\text{or } \tilde{H}_\mu^{b_q, B})$  then p.d.o  $h_{\mu, a}$  on Hankel convolution  $(h_{\mu, a}(\phi \# \psi)) \in \tilde{H}_{\mu, a_k^{*2}, A'_7}(\text{or } H_\mu^{b_q^{*2}, B'_6})$ , where  $A'_7 = (a_0^* / a_1^*)D + B_1$  and  $B'_6 = (b_0^* / b_1^*)C + H_2^6 B^2$ . Similarly we may define  $\phi \# \psi, h_\mu(\phi \# \psi)$  for  $\phi \in H_{\mu, a_k, A}$  and  $\psi \in \tilde{H}_\mu^{b_q, B}$  or  $\psi \in H_{\mu, a_k, A}^{b_q, B}$  and study the p.d.o. on  $\phi \# \psi$ .

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